J. Fluid Mech. (1969), *vol.* 37, *part* 4, *pp.* 773–784
Printed in Great Britain

On the trapping of long-period waves round islands

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(Received 9 November 1968)

The trapping of short-period gravity waves by islands and seamounts has been studied by Chambers (1965) and by Longuet-Higgins (1967). It was shown by the latter that in the absence of rotation, or when the wave frequency σ is large compared with the Coriolis parameter f, these waves cannot be perfectly trapped; some energy must always leak away to infinity. Very long-period oscillations in the presence of a sloping shelf surrounding an island, with $\sigma \ll f$, have been studied by Mysak (1967) and Rhines (1967, 1969). Here perfect trapping is possible. However, as pointed out in Longuet-Higgins (1968), the rotation itself exerts a strong trapping effect not only when $|\sigma| \ll f$, but also whenever $|\sigma| < f$. It seems not to have been noticed that this effect is capable of trapping waves round an island in an ocean of uniform depth, in the absence of any shelf or sloping region offshore.

The existence of such waves is demonstrated for a circular island in § 1 of the present paper. It is shown that the waves exist only when the azimuthal wave-number n is at least 1. The waves always progress round the island in a clockwise sense in the northern hemisphere. At large distances r from the island, the wave amplitude decays exponentially, but this exponential trapping occurs only if the radius a of the island exceeds the critical value $(n(n-1)gh)^{\frac{1}{2}}/f$. When $n = 1$, this critical radius is zero, so that in theory the waves exist for any island of non-zero radius.

The application of these results to the ocean is discussed in § 2. Except possibly for baroclinic motions, it appears that only the waves corresponding to $n = 1$ could exist in fact, and that their frequency would be nearly equal to the inertial frequency f. It is unlikely that f could be regarded as constant over a sufficiently wide area for the model to apply without qualification. Nevertheless, the oscillations may be regarded as the local adjustment of the pressure field to inertial currents in the neighbourhood of the island. It is possible that the peak at about 0·73 c.p.d. in the spectrum of sea-level at Oahu, as found by Miyata & Groves (1968), can be interpreted in this way.

1. Free waves round a circular island

The differential equation satisfied by the surface elevation ζ for long waves in a rotating ocean of uniform depth h is

$$\left(\nabla^2 + \frac{\sigma^2 - f^2}{gh}\right)\zeta = 0, \tag{1.1}$$

where ∇^2 denotes the two-dimensional Laplacian in the horizontal plane, σ is the radian frequency (with $\zeta \propto e^{-i\sigma t}$), and f and g denote the Coriolis parameter and the acceleration of gravity respectively (see, for example, Lamb 1932). Let us take polar co-ordinates (r, θ) in the horizontal plane. Then, as pointed out by Chambers (1965), there exist solutions to (1.1) in the form

$$\zeta = [AJ_n(kr) + BY_n(kr)]\, e^{i(n\theta - \sigma t)}; \tag{1.2}$$

where

$$k^2 = \frac{\sigma^2 - f^2}{gh}, \quad (n = 0, 1, 2, \ldots), \tag{1.3}$$

and J_n and Y_n denote Bessel functions of the first and second kinds. When $\sigma^2 \geqslant f^2$, k is real. Chambers suggested that these expressions represent trapped waves; but, since J_n and Y_n diminish like $r^{-\frac{1}{2}}$ at infinity, the total energy outside a given radius r is finite, and so the waves are not really trapped. (*Virtual* trapping may be achieved by means of a shallow circular sill; see Longuet-Higgins 1967.)

On the other hand, when $\sigma^2 < f^2$, we may write

$$\kappa^2 = \frac{f^2 - \sigma^2}{gh} > 0, \tag{1.4}$$

so that the appropriate solutions are now

$$\zeta = [CI_n(\kappa r) + DK_n(\kappa r)]\, e^{i(n\theta - \sigma\ )}, \tag{1.5}$$

where I_n and K_n represent the modified Bessel functions (see, for example, Olver 1964). Since for large z

$$\left.\begin{aligned} I_n(z) &\sim \left(\frac{1}{2\pi z}\right)^{\frac{1}{2}} e^{z}, \quad |\arg z| < \tfrac{1}{2}\pi, \\ K_n(z) &\sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}, \quad |\arg z| < \tfrac{3}{2}\pi, \end{aligned}\right\} \tag{1.6}$$

it follows that, if the energy outside a given circle is to be finite, we must have $C = 0$, and then, with $D = 1$,

$$\zeta = K_n(\kappa r)\, e^{i(n\theta - \sigma t)}, \tag{1.7}$$

may represent a truly trapped motion, provided that the appropriate boundary conditions for finite r can be satisfied. As pointed out by Longuet-Higgins (1968), the trapping effect is the result of rotation, not of refraction. Since, near $z = 0$,

$$K_n(z) \sim \left\{\begin{aligned} &-\log z, & n = 0, \\ &\frac{2^{n-1}(n-1)!}{z^n}, & n > 0, \end{aligned}\right\} \tag{1.8}$$

the origin $r = 0$ must be excluded.

Suppose we assume that, at the boundary $r = a$ of the island, the radial component of velocity vanishes. This leads to the condition, when $\sigma^2 \neq f^2$,

$$u_r \equiv \frac{-g}{\sigma^2 - f^2}\left(i\sigma \frac{\partial \zeta}{\partial r} - \frac{f}{r}\frac{\partial \zeta}{\partial \theta}\right) = 0, \quad (r = a); \tag{1.9}$$

or, from (1.7),
$$\frac{(\kappa a)\,K_n'(\kappa a)}{K_n(\kappa a)} = \frac{nf}{\sigma}. \tag{1.10}$$
Using the recurrence relation
$$K_n'(z) \equiv \frac{n}{z}K_n(z) - K_{n+1}(z), \tag{1.11}$$
we may write (1.10) in the form
$$\frac{(\kappa a)\,K_{n+1}(\kappa a)}{K_n(\kappa a)} = n\left(1-\frac{f}{\sigma}\right). \tag{1.12}$$

Has the above equation any solutions? When $n = 0$ the right-hand side vanishes. Since K_n has no positive zeros, it follows that no solutions symmetrical about the origin exist. This follows also from the theorem that the total circulation about the boundary of the island is invariant in time; hence, if it is sinusoidal, it must vanish. We are thus left with the case $n \geqslant 1$.

Now, from (1.4), we have
$$(\kappa a)^2 = \epsilon\left(1-\frac{\sigma^2}{f^2}\right), \quad \frac{\sigma^2}{f^2} = 1-\frac{(\kappa a)^2}{\epsilon}, \tag{1.13}$$
where ϵ denotes the non-dimensional parameter
$$\epsilon \equiv \frac{a^2 f^2}{gh}, \tag{1.14}$$
which is always positive. Thus (1.12) may be written in the form
$$F_n(\kappa a) = \frac{-f}{\sigma} = G(\kappa a, \epsilon), \tag{1.15}$$
where
$$\left.\begin{aligned} F_n(\kappa a) &\equiv \frac{(\kappa a)\,K_{n+1}(\kappa a)}{nK_n(\kappa a)} - 1, \\ G(\kappa a, \epsilon) &\equiv \left(1-\frac{(\kappa a)^2}{\epsilon}\right)^{-\frac{1}{2}}. \end{aligned}\right\} \tag{1.16}$$

The variables F_n and G are plotted in figure 1 as functions of $\xi = \frac{1}{2}(\kappa a)^2$, for $n = 1, 2, 3, 4$ and $\epsilon = 0, 2, 6, 12$ (the reasons for this choice of values for ϵ will soon be apparent; see (1.22)). The curves all pass through the point $\xi = 0$, $F_n = G_n = 1$. But it can be seen that whereas F_1 intersects all the G curves (including the curve for $\epsilon = 0$), F_2, F_3 and F_4 intersect only some. It can be shown that the curves for F_n are always concave downwards,† whereas the curves for G are concave upwards; that is to say
$$\frac{d^2F_n}{d\xi^2} < 0, \quad \frac{d^2G}{d\xi^2} > 0. \tag{1.17}$$

Moreover, as $\xi \to \frac{1}{2}\epsilon$, so $G \to \infty$, whereas F_n remains finite. Therefore, apart from $\xi = 0$, any F_n intersects G at one other point, or none, depending on whether
$$dF_n/d\xi \gtrless dG/d\xi \quad \text{at} \quad \xi = 0.$$

† See appendix B.

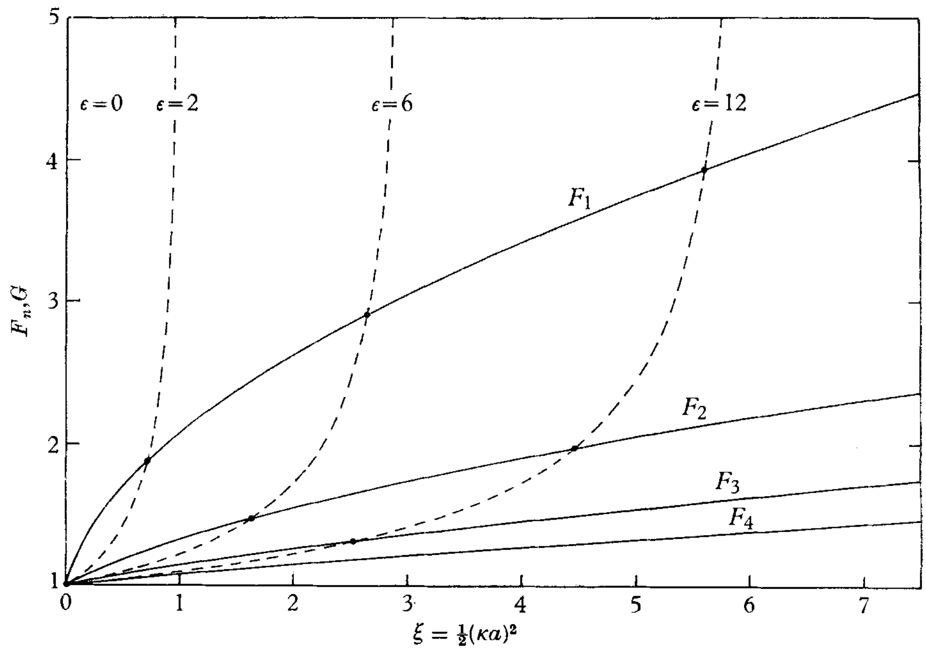

FIGURE 1. Graphs of the functions $F_n(\xi)$ and $G(\xi)$ defined by (1.16). Their intersections give the frequencies of the trapped modes.

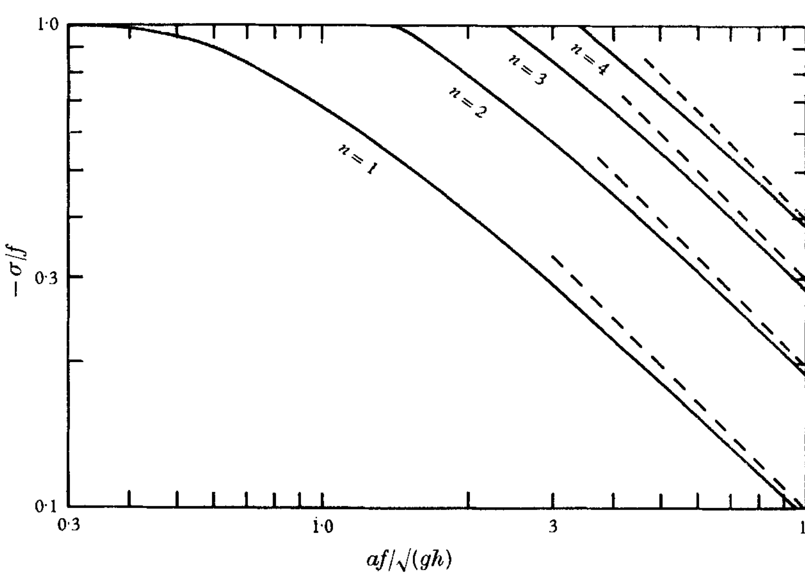

FIGURE 2. Graph of the non-dimensional frequency σ/f, as a function of $af/(gh)^{\frac{1}{2}}$. The broken lines indicate the asymptotic expressions $-\sigma/f = n/\surd\epsilon$, corresponding to Kelvin waves.

Now, from the expansions for $K_n(z)$ when $z \ll 1$ (see Olver 1964), we have

$$\left.\begin{aligned} K_1(z) &= \frac{1}{z}\left[1+\frac{z^2}{4}\ln\left(\frac{z^2}{4}\right)+(\gamma-\tfrac{1}{2})\frac{z^2}{2}+O(z^4\ln z)\right],\\ K_2(z) &= \frac{2}{z^2}\left[1-\frac{z^2}{4}-\frac{z^2}{16}\ln\left(\frac{z^2}{4}\right)+O(z^4)\right],\\ K_n(z) &= \frac{2^{n-1}(n-1)!}{z^n}\left[1-\frac{z^2}{4(n-1)}+\frac{z^4}{32(n-1)(n-2)}-O(z^6)\right] \quad (n \geqslant 3).\end{aligned}\right\} \tag{1.18}$$

Hence

$$\left.\begin{aligned} F_1 &= 1-\xi\ln\left(\frac{\xi}{2}\right)-2\gamma\xi+O(\xi^2\ln\xi),\\ F_2 &= 1+\frac{\xi}{2}+\frac{\xi^2}{2}\ln\left(\frac{\xi}{2}\right)+\frac{\gamma\xi^2}{4}+O(\xi^3\ln\xi),\\ F_n &= 1+\frac{\xi}{n(n-1)}-\frac{\xi^2}{2n(n-1)^2(n-2)}+O(\xi^3\ln\xi) \quad (n \geqslant 3).\end{aligned}\right\} \tag{1.19}$$

So, as $\xi \to 0$,

$$\frac{dF_1}{d\xi} \to \infty, \quad \frac{dF_n}{d\xi} \to \frac{1}{n(n-1)} \quad (n \geqslant 1). \tag{1.20}$$

On the other hand, when $\xi \to 0$, we have

$$G = 1+\frac{\xi}{\epsilon}+\frac{3\xi^2}{2\epsilon^2}+O(\xi^3), \qquad \frac{dG}{d\xi} \to \frac{1}{\epsilon}. \tag{1.21}$$

It follows that when $n = 1$ equation (1.15) has always just one positive root, but that when $n = 2, 3, 4, \ldots$ there is one positive root if

$$\epsilon > n(n-1); \tag{1.22}$$

otherwise there is none.

The condition (1.22) can also be written

$$a > (n(n-1)gh)^{\frac{1}{2}}/f, \tag{1.23}$$

which shows that for given values of f, g, h and n there is a minimum radius a for the existence of trapped modes. If the radius is less than this critical value, the waves cannot 'round the corner'.

The lowest mode ($n = 1$) may be included in the above statement if we allow the critical radius to be zero.

For large values of n the transverse wave-number at the edge of the island is approximately $n/a = m$, say, and in that case the condition (1.23) may be written in the form

$$\frac{f^2}{m^2gh} > \left(1-\frac{1}{n}\right). \tag{1.24}$$

To obtain the frequency σ of the basic modes, for any given value of ϵ and n, we may use figure 1 to find the corresponding value of G, $= -f/\sigma$, and then take the inverse. In this way we have derived figure 2.

When (κa) is large compared to n, the asymptotic formulae (1.6) will apply, and we shall have

$$F_n \sim \frac{\kappa a}{n}. \tag{1.25}$$

Since also G must be large, it follows that

$$\kappa a \sim \sqrt{\epsilon}; \tag{1.26}$$

and so

$$-\frac{f}{\sigma} = F_n \sim \frac{\sqrt{\epsilon}}{n} = \frac{af}{n(gh)^{\frac{1}{2}}}. \tag{1.27}$$

Hence, writing $m = n/a$ for the wave-number at the boundary, we have

$$\frac{\sigma}{m} \sim -(gh)^{\frac{1}{2}}. \tag{1.28}$$

In other words, the velocity of propagation of the waves is equal to $(gh)^{\frac{1}{2}}$. The surface elevation (1.7) is given approximately by

$$\zeta \propto \left(\frac{\pi}{2(\kappa a)}\right)^{\frac{1}{2}} \exp\{-\kappa(r-a)+i(ms-\sigma t)\}, \tag{1.29}$$

where $s = a\theta$, the distance along the perimeter. Since $\kappa \sim f/(gh)^{\frac{1}{2}}$, this shows that the waves are then approximately Kelvin waves, propagated clockwise round the island (in the northern hemisphere). The asymptotic expressions $-\sigma/f = n/\sqrt{\epsilon}$, corresponding to Kelvin waves, are shown in figure 2 by broken lines.

It remains to investigate the special case when $\sigma = \pm f$. From figure 2 it can be seen that, even when $\epsilon > 0$, there exists a root of (1.15) given by

$$\kappa a = 0, \quad \sigma = -f. \tag{1.30}$$

Does this correspond to a possible trapped motion?

In appendix A it is shown that in this case there can be a trapped wave only when

$$\epsilon = n(n-1), \quad (n \geqslant 2), \tag{1.31}$$

i.e. in the special circumstances when the radius a of the island equals the critical radius. In all other cases no trapped wave exists. In this sense figure 2 is a complete diagram. We conclude that the difference between the wave frequency and the inertial frequency is an essential feature in the trapping of the wave motion, except in the special case (1.31).†

2. Applications to the ocean

For most oceanographic applications ϵ is a small quantity. For instance, corresponding to the island of Oahu (latitude 21° 30′ N) we may take

$$\left.\begin{aligned} a &= 100\,\mathrm{km}, & f &= 5\times 10^{-5}\,\mathrm{rad/s}, \\ g &= 10^{-2}\,\mathrm{km/s^2}, & h &= 3\,\mathrm{km}, \end{aligned}\right\} \tag{2.1}$$

giving $\epsilon = 0{\cdot}83 \times 10^{-3}$. For this reason it appears that when $n \geqslant 2$ the condition $\epsilon > n(n-1)$ cannot be satisfied by the barotropic motions. Hence only the lowest mode ($n = 1$) can exist.

It is clear from figure 1 that if ϵ is small, then for the mode $n = 1$ the quantity $\xi = \frac{1}{2}(\kappa a)^2$ is small also. Moreover, since G is close to unity, (ξ/ϵ) must be a small

† This conclusion may have to be modified when non-linear effects are taken into account.

quantity. Equating the expansions for F_1 and G in (1.19) and (1.21) we obtain in fact

$$-\ln\left(\frac{\xi}{2}\right) \doteqdot \frac{1}{\epsilon}; \tag{2.2}$$

so

$$(\kappa a) = (2\xi)^{\frac{1}{2}} \doteqdot 2\,e^{-1/2\epsilon}. \tag{2.3}$$

This being extremely small, it follows that the region of exponential trapping, which is where $\kappa r \gg 1$, would be at unrealistically large values of (r/a)—so large that f could not in practice be assumed to be uniform over the whole region.

Nevertheless, the solution (1.7) corresponding to $n = 1$, namely

$$\zeta = K_1(\kappa r)\,e^{i(\theta-\sigma t)}, \tag{2.4}$$

still represents a solution which is valid locally near the island. From (1.18) we see that the first terms in its asymptotic expansion are given by

$$\zeta \sim \left[\frac{1}{\kappa r}+\frac{\kappa r}{2}\ln\left(\frac{\kappa r}{2}\right)\right]e^{i(\theta-\sigma t)}, \quad (a < r \ll \kappa^{-1}), \tag{2.5}$$

and from (1.13)

$$\sigma \sim -f\left(1-\frac{(\kappa a)^2}{2\epsilon}\right)^{\frac{1}{2}} \sim -f. \tag{2.6}$$

Thus the frequency is approximately equal to the inertial frequency, and the motion progresses clockwise round the island, in the northern hemisphere. The radial and tangential components of velocity are given by

$$\left.\begin{aligned} u_r &\sim \frac{-if}{\kappa h}\left[\frac{1}{2\epsilon}\frac{a^2}{r^2}+\ln\left(\frac{\kappa r}{2}\right)\right]e^{i(\theta-\sigma t)},\\ u_\theta &\sim \frac{-f}{\kappa h}\left[\frac{1}{2\epsilon}\frac{a^2}{r^2}-\ln\left(\frac{\kappa r}{2}\right)\right]e^{i(\theta-\sigma t)}. \end{aligned}\right\} \tag{2.7}$$

When $r = a$, the radial velocity u_r vanishes, by equation (2.2). Both terms in the expression for u_r diminish at first, the term in a^2/r^2 rapidly and the term in $\ln(\kappa r)$ more gradually. Hence there is a limited concentration of energy in the neighbourhood of the island itself.

The record of sea-level at Mokuoloe and Honolulu, on the island of Oahu, shows significant peaks in the coherence spectrum at frequencies of 0·73, 0·50, 0·33 and 0·23 c.p.d. (Miyata & Groves 1968). The inertial frequency at that latitude (21° 30′ N) is 0·73 c.p.d. Hence it seems reasonable to identify this peak as the local aspect of inertial oscillations in the neighbourhood of the island. The phase difference of 120° between Mokuoloe and Honolulu is also consistent with this interpretation. A further discussion of the spectrum of sea-level at Oahu is given in another paper (in preparation).

Consider, on the other hand, baroclinic motions. These are subject to an exactly similar analysis, except that the vertical displacement ζ is now a function of the vertical co-ordinate. Also, ϵ is to be replaced by a larger parameter ϵ', which is of order $(\rho/\Delta\rho)\,\epsilon$, where ρ denotes the mean density and $\Delta\rho$ the density difference between top and bottom of the ocean. Since $(\rho/\Delta\rho)$ may be of order 10^2, it is possible, if a is sufficiently large, for ϵ' to be of order unity. In that case it appears that the solutions of § 1 may well be applicable.

3. Conclusions

A circular island in a rotating ocean of uniform depth h and constant Coriolis parameter f can effectively trap energy in the form of progressive waves, but only if f is constant over a very wide area. The frequency of the trapped modes is always less than the inertial frequency, and the waves progress round the island in the same direction as inertial oscillations, that is to say clockwise in the northern hemisphere.

If the azimuthal wave-number $n = 1$, trapping is always possible in theory. If on the other hand $n \geqslant 2$, trapping is possible only if the radius of the island exceeds $(n(n-1)gh)^{\frac{1}{2}}/f$. For barotropic motions (but not baroclinic motions) in the sea, this quantity is generally so large that trapping is impossible.

The peak at 0·73 c.p.d. in the sea-level spectrum at Oahu probably represents the forced response of the island to currents at nearly the inertial frequency. This hypothesis also accounts for the observed phase difference between Mokuoloe and Honolulu, assuming that some of the energy is propagated round the Hawaiian ridge as a whole. It would be interesting to examine records of sea-level at other island stations for evidence of similar oscillations.

I am indebted to Dr Gordon W. Groves for stimulating correspondence on this subject. The present research has been supported under N.S.F. Grant GA-1452.

Appendix A. Inertial oscillations

Suppose first that the frequency σ is equal to $-f$, so that u, v and ζ are proportional to e^{ift}. The linearized equations of motion become

$$\left.\begin{aligned} if(u+iv) &= -g\frac{\partial\zeta}{\partial x}, \\ if(v-iu) &= -g\frac{\partial\zeta}{\partial y}. \end{aligned}\right\} \tag{A 1}$$

Hence we have
$$(u+iv) = \frac{ig}{f}\frac{\partial\zeta}{\partial x} = -\frac{g}{f}\frac{\partial\zeta}{\partial y}, \tag{A 2}$$

and so
$$\frac{\partial\zeta}{\partial x} = -\frac{\partial\zeta}{\partial(iy)}, \quad \nabla^2\zeta = 0, \tag{A 3}$$

which is the special form of (1.1) when $\sigma^2 = f^2$. This can be satisfied by taking ζ to be an analytic function of $z^* = (x-iy)$, so that

$$(u+iv) = \frac{ig}{f}\frac{\partial\zeta}{\partial z^*}. \tag{A 4}$$

The equation of continuity, when $\sigma = -f$, becomes

$$h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = -if\zeta. \tag{A 5}$$

To see that this can also be satisfied, write

$$(u-iv) = Z. \tag{A 6}$$

Then, from (A 2) and (A 6),

$$u = \frac{1}{2}\left(\frac{ig}{f}\frac{\partial\zeta}{\partial x} + Z\right), \quad v = \frac{i}{2}\left(\frac{g}{f}\frac{\partial\zeta}{\partial y} + Z\right). \tag{A 7}$$

Substituting in (A 5), we have $\dfrac{\partial Z}{\partial x} + i\dfrac{\partial Z}{\partial y} = -\dfrac{2if}{h}\zeta,$ (A 8)

of which the solution is $Z = -\dfrac{if}{h}\displaystyle\int \zeta\, dz^* + F(z)\, e^{ift},$ (A 9)

where F is an arbitrary analytic function of $z = (x+iy)$.

The radial velocity is given by

$$u_r = u\cos\theta + v\sin\theta, \tag{A 10}$$

i.e.

$$2u_r = (u-iv)\, e^{i\theta} + (u+iv)\, e^{-i\theta}, \tag{A 11}$$

$$= Z\, e^{i\theta} + \frac{ig}{f}\frac{d\zeta}{dz^*} e^{-i\theta}. \tag{A 12}$$

This expression is to vanish on the boundary $r = a$.

From (A 4) the appropriate form for ζ bounded at infinity is

$$\zeta = \frac{1}{r^n} e^{i(n\theta - \sigma t)} = \frac{e^{ift}}{z^{*n}}. \tag{A 13}$$

When $n = 1$, we have $\zeta = \dfrac{e^{ift}}{z^*}$; (A 14)

so from (A 9) $Z = \left[\dfrac{if}{h}\log z^* + F(z)\right] e^{ift}.$ (A 15)

The condition that u_r shall vanish at $r = a$ now becomes

$$\frac{if}{h}(\log a - i\theta) + F(a\, e^{i\theta}) - \frac{ig}{fa^2} = 0, \tag{A 16}$$

or

$$F(a\, e^{i\theta}) = \frac{ig}{fa^2}(1 - \epsilon \log a) - \frac{f}{h}\theta. \tag{A 17}$$

This can only be satisfied by an expression of the form

$$F(z) \equiv \frac{f}{ih}\log z + \text{const.} \tag{A 18}$$

Since $\log z \to \infty$ as $r \to \infty$, this does not represent a trapped motion.

On the other hand, when $n \geqslant 2$ we have

$$Z = \left[\frac{if}{(n-1)h}\frac{1}{z^{*(n-1)}} + F(z)\right] e^{ift}. \tag{A 19}$$

The condition that u_r shall vanish at $r = a$ gives

$$\frac{if}{(n-1)h}\frac{e^{in\theta}}{a^{n-1}} + e^{i\theta}F(a\, e^{i\theta}) - \frac{ign}{f}\frac{e^{in\theta}}{a^{n+1}} = 0. \tag{A 20}$$

So
$$e^{i\theta}F(a\,e^{i\theta}) = \frac{ign}{fa^{n+1}}\left(1-\frac{\epsilon}{n(n-1)}\right)e^{in\theta}. \tag{A 21}$$

Now, if the coefficient of $e^{in\theta}$ does not vanish, then $F(z)$ is proportional to $z^{(n-1)}$, $= r^{n-1}\,e^{i(n-1)\theta}$, which tends to infinity as $r \to \infty$. So for a bounded solution we must have
$$\left(1-\frac{\epsilon}{n(n-1)}\right) = 0, \qquad \epsilon = n(n-1). \tag{A 22}$$

In other words, the radius a must be exactly equal to the critical radius $(n(n-1)\,gh)^{\frac{1}{2}}/f$. It would be interesting to verify this conclusion experimentally.

Appendix B. The sign of $d^2F/d\xi^2$

Let us define
$$y \equiv -\frac{z}{K}\frac{dK}{dz} = nF_n, \tag{B 1}$$

where for brevity we have written $K_n = K$. Thus K is the solution of Bessel's modified equation:
$$\left(z\frac{d}{dz}\right)\left(z\frac{dK}{dz}\right) = (n^2+z^2)\,K, \tag{B 2}$$

which tends to zero as $z \to \infty$. From the asymptotic formulae (1.6) it follows that as $z \to \infty$,
$$K > 0, \quad \left(z\frac{dK}{dz}\right) > 0. \tag{B 3}$$

From (B 2) it then follows that $K > 0$ and $(z\,dK/dz) < 0$ for all positive z. Hence $y > 0$ over the range of interest. What we have now to show is that, for all positive z,
$$\frac{d^2y}{dx^2} < 0, \tag{B 4}$$

where $x = z^2 = 2\xi$ (see § 1).

Now, from the definition of y (equation (B 1)), it follows that
$$z\frac{dy}{dz} = \frac{1}{K^2}\left(z\frac{dK}{dz}\right)^2 - \frac{1}{K}\left(z\frac{d}{dz}\right)\left(z\frac{dK}{dz}\right); \tag{B 5}$$

and, on substituting from the differential equation (B 2), we obtain simply
$$z\frac{dy}{dz} = y^2-(n^2+z^2), \tag{B 6}$$

a first-order, non-linear equation for y. Since $d/dz = 2z\,d/dx$, we have then
$$\frac{dy}{dx} = \frac{y^2-n^2-x}{2x}. \tag{B 7}$$

Differentiating again, this time with respect to x, we obtain
$$\frac{d^2y}{dx^2} = \frac{y}{x}\frac{dy}{dx} - \frac{y^2-n^2}{2x^2}, \tag{B 8}$$

and, on substitution from (B 7),

$$\frac{d^2y}{dx^2} = \frac{1}{2x^2}[(y-1)(y^2-n^2-x)-x]. \tag{B 9}$$

Therefore it is sufficient to show that

$$P < 0, \tag{B 10}$$

where $$P \equiv (y-1)(y^2-n^2-x)-x. \tag{B 11}$$

To do this we differentiate (B 11) once more:

$$\begin{aligned}\frac{dP}{dx} &= \frac{dy}{dx}(y^2-n^2-x)+(y-1)\left(2y\frac{dy}{dx}-1\right)-1 \\ &= 2x\left(\frac{dy}{dx}\right)^2+y\left(2(y-1)\frac{dy}{dx}-1\right) \\ &= 2x\left(\frac{dy}{dx}\right)^2+\frac{y}{x}P. \end{aligned} \tag{B 12}$$

So for P we have the first-order differential equation

$$\frac{x}{y}\frac{dP}{dx}-P = Q, \tag{B 13}$$

where $$Q = \frac{2x^2}{y}\left(\frac{dy}{dx}\right)^2 \geqslant 0, \tag{B 14}$$

since $y > 0$, as shown earlier. Now since y is always positive we may define a new independent variable χ by the equation

$$\chi = \int_1^x \frac{y}{x}dx, \tag{B 15}$$

so that (B 13) becomes simply

$$\left(\frac{d}{d\chi}-1\right)P = Q. \tag{B 16}$$

The solution of this equation is

$$P\,e^{-\chi} = -\int_\chi^\infty Q\,e^{-\chi}\,d\chi + \text{const.} \tag{B 17}$$

But, as $x \to \infty$, we have from the asymptotic formulae for $K_n(z)$

$$y \sim x^{\frac{1}{2}}, \quad P \sim -\tfrac{1}{2}x^{\frac{1}{2}}, \quad \chi \sim 2x^{\frac{1}{2}}. \tag{B 18}$$

Hence, on taking the limit as $x \to \infty$ in equation (B 17), we see that the constant of integration must vanish. Therefore we have

$$P = -e^\chi \int_\chi^\infty Q\,e^{-\chi}\,d\chi. \tag{B 19}$$

Since Q is non-negative, it follows that P must be strictly negative. This proves the result.

REFERENCES

CHAMBERS, L. G. 1965 On long waves on a rotating earth. *J. Fluid Mech.* **22**, 209–216.

LAMB, H. 1932 *Hydrodynamics* (sixth edition). Cambridge University Press.

LONGUET-HIGGINS, M. S. 1967 On the trapping of wave energy round islands. *J. Fluid Mech.* **29**, 781–821.

LONGUET-HIGGINS, M. S. 1968 On the trapping of waves along a discontinuity of depth in a rotating ocean. *J. Fluid Mech.* **31**, 417–434.

LONGUET-HIGGINS, M. S. 1969 On the spectrum of sea level at Oahu. (To be published.)

MIYATA, M. & GROVES, G. W. 1968 Note on sea-level observations at two nearby stations. *J. Geophys. Res.* **73**, 3965–3967.

MYSAK, L. A. 1967 On the theory of continental shelf waves. *J. Mar. Res.* **25**, 205–227.

OLVER, F. W. J. 1964 Bessel functions of integer order. In *Handbook of Mathematical Functions* (eds. M. Abramowitz and I. A. Stegun), ch. 9. Washington, D. C.: National Bureau of Standards.

RHINES, P. B. 1967 The influence of bottom topography on long-period waves in the ocean. Ph.D. Thesis, Cambridge University.

RHINES, P. B. 1969 *J. Fluid Mech.* (To be published.)